From Heat Equation to Quadratic Forms: High-Dimensional Probability as Curved Diffusion

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Roadmap

1. Recap: N-dimensional heat equation as a Gaussian random vector
2. Anisotropic diffusion and the diffusion tensor D
3. Quadratic forms $x^\top Ax$ as curved probability spaces
4. Chapter 6 toolkit: decoupling, symmetrization, contraction, HW
5. Curvature $\Rightarrow$ concentration: Riemannian manifolds
6. Heat flow, Fokker–Planck, and Ricci flow as one diffusion grammar

Step 1: N-Dimensional Heat Equation (Recap)

$$\partial_t u = D\Delta u = D\sum_{i=1}^{n} \partial^2_{x_i x_i} u, \qquad u(x,0) = \delta_0(x),$$

$$u(x,t) = \frac{1}{(4\pi D t)^{n/2}} \exp\left(-\frac{\|x\|^2}{4Dt}\right).$$

- Fundamental solution $=$ n-D Gaussian kernel.
- Already a probability density if we normalize: $\int_{\mathbb{R}^n} u(x,t)\,dx = 1$.

Step 2: Heat Kernel as Gaussian Random Vector

Rename $u(x,t) \to p(x,t)$. Then

$$p(x,t) = \frac{1}{(4\pi Dt)^{n/2}} \exp\left(-\frac{\|x\|^2}{4Dt}\right) \quad \Longleftrightarrow \quad X_t \sim N\big(0, \Sigma_t\big), \; \Sigma_t = 2Dt\, I_n.$$

- Each time-slice of the heat field is a Gaussian random vector.
- Diffusion coefficient D is just the variance growth rate.
- Heat PDE and Brownian SDE describe the same process:

$$\partial_t u = D\Delta u \quad \Longleftrightarrow \quad dX_t = \sqrt{2D}\, dW_t.$$

Step 3: Anisotropic Diffusion and the Tensor D

General anisotropic diffusion:

$$\partial_t u = \nabla \cdot (D \nabla u), \qquad D = D^\top > 0,$$

has solution

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}\sqrt{\det D}} \exp\left(-\frac{1}{4t} x^\top D^{-1} x \right), \quad \Sigma_t = 2tD.$$

- Eigenpairs (λ_i, q_i) of D: principal diffusion directions and variances.
- Level sets $x^\top \Sigma_t^{-1} x = c$: ellipsoids (tilt = correlation).

PDE vs Random Vector View

PDE / Heat	**Random Vector**
$\partial_t u = \nabla \cdot (D \nabla u)$	$X_t \sim N(0, \Sigma_t)$
Diffusion tensor D	Covariance $\Sigma_t = 2tD$
Isotropic: $D = DI_n$	i.i.d. coordinates
Anisotropic SPD D	correlated Gaussian / ellipsoid

- Heat equation $\Rightarrow$ law of motion for Gaussian random vectors.
- This is already *high-dimensional probability in disguise*.

Step 4: Quadratic Forms as Energy

Now move to Chapter 6: consider

$$Q_A(x) = x^\top A x, \qquad A \in \mathbb{R}^{n\times n} \text{ symmetric.}$$

Interpretations:

- Random energy of a vector X: $Q_A(X) = X^\top A X$.
- In coordinates where $A = Q\Lambda Q^\top$,

$$Q_A(x) = \sum_i \lambda_i (q_i^\top x)^2.$$

  Eigenvalues λ_i = curvature along principal directions.
- Compare with anisotropic heat kernel exponent $\frac{1}{4t} x^\top D^{-1} x$: here $A \sim D^{-1}$ encodes geometry.

Heat Tensor D vs Curvature Matrix A

Heat / Diffusion	**Quadratic Form / Chapter 6**
SPD tensor D	SPD matrix A
Kernel exponent $\frac{1}{4t}x^\top D^{-1}x$	Energy $x^\top Ax$
Principal axes = eigenvectors of D	Principal axes = eigenvectors of A
Variances $\propto \lambda_i(D)$	Curvature weights $\propto \lambda_i(A)$

- Quadratic forms are *curved probability spaces*: A is the metric.
- Chapter 6 studies how randomness behaves under this curvature.

Chapter 6 Tools in One Picture

For independent mean-zero X_i:

- **Decoupling:** replace $X^\top AX$ by $X^\top AX'$ (bilinear chaos).
- **Symmetrization:** introduce signs ε_i or differences $X_i - X_i'$.
- **Contraction:** Lipschitz maps shrink chaos norms.
- **Hanson–Wright:** curvature bounds $\|A\|_F, \|A\|$ give sub-Gaussian tails:

$$(|X^\top AX - [X^\top AX]| > t) \le 2\exp\left[-c\min\left(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|}\right)\right].$$

Think of this as a *discrete heat-kernel bound for random energy*.

Geometric Reading of Hanson–Wright

- $\|A\|_F^2 = \sum \lambda_i^2$: total curvature (sum of squared principal curvatures).
- $\|A\|$: maximal curvature direction.
- Hanson–Wright says:

"Random energy concentrates at a rate set by the curvature of A."

- Compare with your Ch. 5 note: Riemannian manifolds with positive Ricci curvature satisfy

$$\|f(X) - f(X)\|_{\psi_2} \leq \frac{C\|f\|}{\sqrt{c(M)}},$$

where $c(M)$ is a Ricci lower bound.

Same Pattern: Curvature $\Rightarrow$ Concentration

Euclidean / HDP	**Riemannian / Geometry**
Matrix A	Metric g
HW: $\|A\|_F, \|A\|$ bound tails of $X^\top AX$	Ricci lower bound $c(M)$ bounds tails of $f(X)$
Quadratic form	Distance function on M

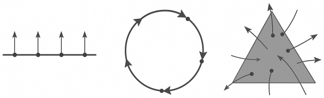
- Chapter 6 is the *flat* version of curvature–concentration.
- The geometric statement in 5.2.4 is the *curved* version.

Diffusion Trilogy

Flat Heat

$$\partial_t \rho = \Delta \rho$$

- Euclidean metric fixed.
- Gaussian kernel.



(a) Constant linear diffusion

(b) Constant curved diffusion

(c) Nonconstant nonlinear diffusion

Curved / Fokker–Planck

$$\partial_t \rho = (\rho \nabla (\log \rho + V)).$$

- Fixed curved geometry or potential V.
- Bakry–Émery: $\mathrm{Ric} + \nabla^2 V \geq \rho g \Rightarrow$ concentration.

Ricci Flow

$$\partial_t g = -2\,\mathrm{Ric}(g).$$

- Metric itself diffuses.
- Perelman's entropy $=$ geometric HW.

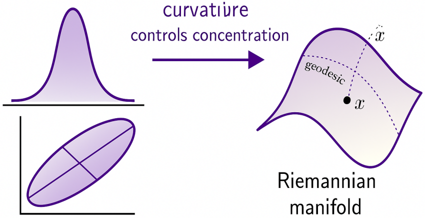
Flat (Chapter 6) $\Rightarrow$ Curved $\Rightarrow$ Self-curving

From Chapter 6 to Ricci Flow

What we have so far:

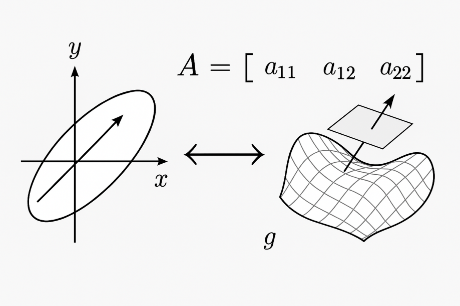
- Heat equation $\Rightarrow$ Gaussian random vector
- Diffusion tensor $D \Rightarrow$ covariance / geometry
- Quadratic forms $x^\top Ax \Rightarrow$ curvature operator
- HW: curvature of A controls concentration
- Bakry–Émery: Ricci curvature controls concentration on M
- Ricci flow: metric $g(t)$ diffuses its own curvature

1. **Flat case:** HW controls random energy under fixed A
2. **Curved case:** $\mathrm{Ric} + \nabla^2 V \geq \rho g$ controls Lipschitz tails
3. **Evolving metric:** $A \rightsquigarrow g(t)$ + conjugate heat density
4. **Non-collapse:** HW-style concentration $\Rightarrow$ κ-volume bounds
5. **Solitons & classification:** blow-up limits are shrinking solitons; surgeries preserve entropy; only S^3 survives simply-connectedness



HDP–Perelman Dictionary (with Geometry)

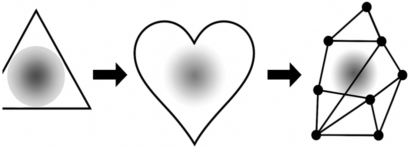
HDP (Chapter 6)	Ricci–Perelman
Matrix A	Metric g_{ij}
$\|A\|_F$, $\|A\|$	scalar / sectional curvature
Chaos, decoupling	linearization, tangent bundle
Contraction	geodesic contraction under $\mathrm{Ric} \geq \rho g$
HW inequality	Bakry–Émery concentration
Sub-Gaussian tails	heat kernel upper bounds
Entropy $S[\rho]$	Perelman's $[g, f]$
HW equality case	shrinking soliton



Where This Talk is Going (Research Direction)

Our synthesis:

- N-D heat eq: diffusion = Gaussian probability
- Chapter 6: quadratic forms = curved probability space
- Curvature $\Rightarrow$ concentration in both settings
- Ricci flow: curvature diffuses itself



Diffusion along curved space or network

My project

1. HDP version of Perelman's entropy $_{\mathrm{HDP}}$
2. HW-based proof of κ-noncollapse
3. HDP reformulation of the Ricci flow route to Poincaré

Putting It All Together

- N-D heat equation already gave us a PDE picture of Gaussian random vectors.
- Chapter 6 takes the same structure and generalizes it to arbitrary quadratic energy $x^\top Ax$.
- The geometric section on Riemannian manifolds shows the same pattern in curved spaces.
- Ricci flow is the ultimate case: the *metric* is the random object evolving under its own curvature.

Takeaway

High-dimensional probability, heat diffusion, and curvature flow are different presentations of one structure: *curvature controls how randomness and energy spread in space*.

Questions / Discussion

How would *you* use Chapter 6 tools on a curved manifold?