The N-Dimensional Heat Equation

a Model for Random Vectors

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Heat Equation and the Gaussian Kernel

Step 1 – Start with the n-dimensional heat equation

$$\frac{\partial u}{\partial t} = \underbrace{D\,\Delta u}_{\text{diffusion term}} = D\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}, \qquad u(x,0) = \delta(x), \quad x \in \mathbb{R}^n.$$

The fundamental solution ("heat kernel") is

$$u(x,t) = \frac{1}{(4\pi Dt)^{n/2}}\exp\left[-\frac{|x|^2}{4Dt}\right].$$

This Gaussian kernel represents the probability density of a Brownian particle initially concentrated at the origin, spreading over time with variance $2Dt$. In other words, the heat equation is the probabilistic law of diffusion.

Heat Equation as a Gaussian Random Process

Step 2 – Interpret $u(x,t)$ as a probability density
If you rename $u(x,t) \to p(x,t)$ and require

$$\int_{\mathbb{R}^n} p(x,t)\,dx = 1,$$

then p is the **joint probability density** of a random vector

$$X_t \sim \mathcal{N}(0, \Sigma_t), \qquad \Sigma_t = 2Dt\, I_n.$$

So each snapshot of the heat field is **the distribution of an n-dimensional Gaussian random vector** whose covariance grows linearly in time.

*Interpretation: The diffusion coefficient D acts as a variance-scaling factor. As time evolves, probability mass spreads isotropically in $\mathbb{R}^n$, and the heat equation becomes equivalent to the Fokker–Planck equation for Brownian motion. This reveals that **heat flow and Gaussian diffusion are the same process** viewed in PDE vs. probabilistic form.*

How the Heat Equation Encodes Random-Vector Geometry

Step 3 – Random-vector structure revealed

- $x_1, \ldots, x_n \to$ components $X_1, \ldots, X_n$
- Diffusion $D \to$ variance growth rate
- Tensor $D_{ij} \to$ covariance entries $\mathrm{Cov}(X_i, X_j)$
- Isotropy (DI_n) $\to$ i.i.d. coordinates (spherical covariance)
- Anisotropy (general D) $\to$ correlated coords / oriented covariance ellipsoid
- Solution $u(x,t) \to$ joint density $p_X(x)$

Heat flow = Gaussian diffusion: the PDE updates the covariance structure over time.

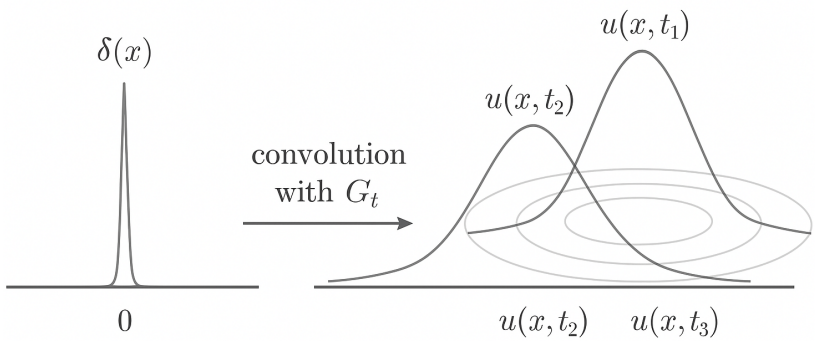
Step 4 – From Heat Flow to Brownian Motion

The heat equation and Brownian motion describe the same phenomenon — just from different perspectives. The PDE shows how heat spreads out continuously in space; the SDE shows how individual particles wander randomly. Their connection is that the probability density of those random particles is the solution to the heat equation.

$$\frac{\partial u}{\partial t} = D\Delta u \quad \Longleftrightarrow \quad dX_t = \sqrt{2D}\, dW_t$$

Interpretation:

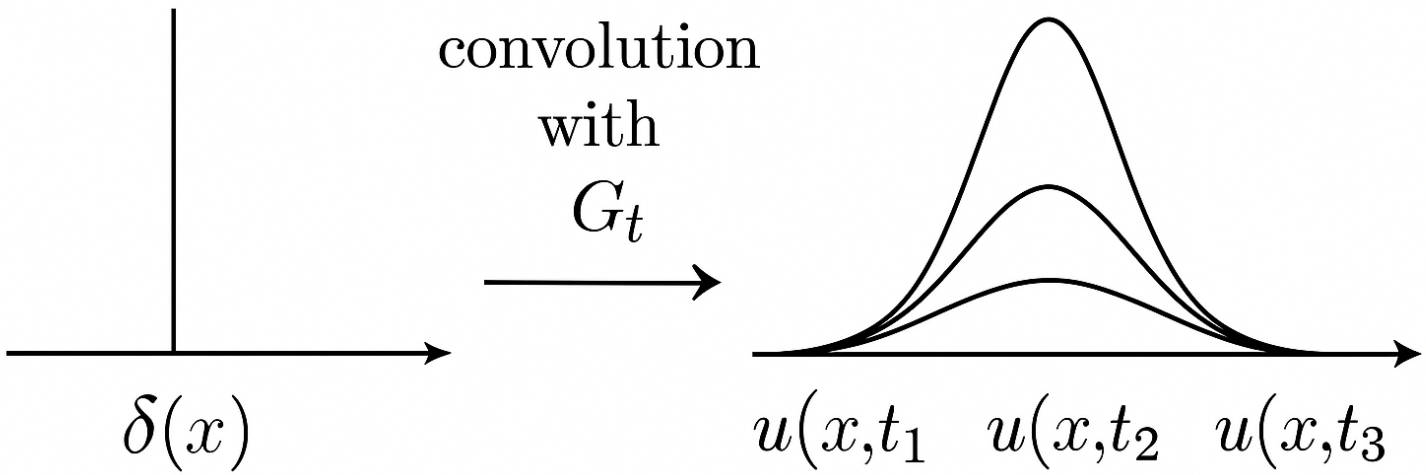
The PDE describes density evolution; the SDE describes particle trajectories.
A Brownian particle's position X_t has the same density $u(x, t)$.



Step 5 – Covariance Growth and Isotropy

$$X_t \sim \mathcal{N}(0, \Sigma_t), \qquad \Sigma_t = 2DtI_n$$

- The covariance matrix scales linearly with time.
- Each coordinate evolves independently, identically.
- Heat diffusion = isotropic Gaussian spreading.



Step 6 – Anisotropic (correlated) case

If diffusion differs by direction or includes cross-terms,

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u), \qquad D = D^\top > 0,$$

then

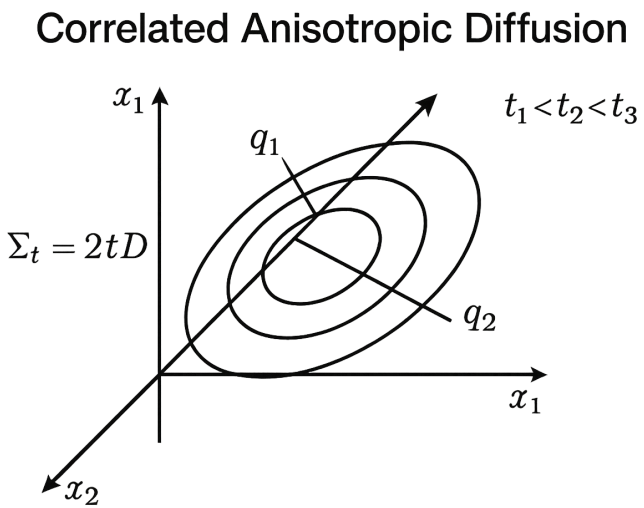
$$u(x,t) = \frac{1}{(4\pi t)^{n/2} |D|^{1/2}} \exp\left[-\frac{1}{4t} x^\top D^{-1} x \right], \qquad \Sigma_t = 2tD.$$

- Eigenvectors of $D \Rightarrow$ principal diffusion directions.
- Eigenvalues $\lambda_i \Rightarrow$ variances along those directions ($\sigma_i^2 = 2t\lambda_i$).
- Level sets $x^\top \Sigma_t^{-1} x = c$ form **ellipsoids**: the geometric signature of correlated diffusion.

Anisotropic diffusion generalizes the Gaussian kernel: each eigenvalue–eigenvector pair defines an axis and rate of spread.

Anisotropic (correlated) diffusion

$$\partial_t u = \nabla\cdot(D\nabla u), \qquad u(x,t) = \frac{1}{(4\pi t)^{n/2}|D|^{1/2}} \exp\left[-\frac{1}{4t}x^\top D^{-1}x\right], \qquad \Sigma_t = 2tD.$$



- **Eigenvectors of** D**:** principal directions (q_1, q_2).
- **Eigenvalues** λ_i**:** variances $\sigma_i^2 = 2t\lambda_i$.
- **Level sets** $x^\top \Sigma_t^{-1} x = c$**:** ellipses (tilt $\Rightarrow$ correlation).

Anisotropy couples diffusion directions — each eigenvector–eigenvalue pair defines both spread and orientation.

Properties of the Diffusion Tensor D

1. D is symmetric positive definite (SPD):

$$D = D^\top, \qquad x^\top D x > 0 \text{ for all nonzero } x.$$

- Ensures **real, positive eigenvalues** $\lambda_i > 0$.
- Guarantees **orthogonal eigenvectors** q_i.
- Allows spectral decomposition: $D = Q\Lambda Q^\top$.

2. Geometric meaning:

- Each eigenpair (λ_i, q_i) defines a **principal direction** q_i with diffusion rate $\sigma_i^2 = 2t\lambda_i$.
- Level sets $x^\top \Sigma_t^{-1} x = c$ ($\Sigma_t = 2tD$) are ellipsoids with semi-axes $\sqrt{2t\lambda_i}$.

3. Physical interpretation:

- Complex eigenvalues $\Rightarrow$ non-physical (rotational or unstable) diffusion.
- SPD form preserves **pure dissipation and positivity of entropy**.

Hence, all physically valid diffusion tensors have real, positive eigenvalues — each describing an independent, stable diffusion axis.

The Heat Equation as a Law of Motion (for Probability)

Every governing law in physics describes *how something evolves in time.*

Mechanics	$\frac{d^2x}{dt^2} = \frac{F}{m}$
Quantum Mechanics	$i\hbar \frac{\partial \psi}{\partial t} = H\psi$
Statistical Mechanics / Diffusion	$\frac{\partial u}{\partial t} = D\,\Delta u$

Microscopic law (SDE): $dX_t = \sqrt{2D}\, dW_t$

Macroscopic law (PDE): $\partial_t u = D\Delta u$

Deterministic laws govern trajectories.
Stochastic laws govern distributions.
Both are expressions of the same principle: ***time evolution through differential structure.***

From Deterministic to Statistical Laws of Motion

All physical laws describe how information spreads in time.

Newtonian law	Motion of a *particle*: $x(t)$
Schrödinger law	Motion of a *wavefunction*: $\psi(x,t)$
Heat law	Motion of a *probability density*: $u(x,t)$

Deterministic system: $dX_t = v(X_t)\,dt$

Stochastic system: $dX_t = \sqrt{2D}\,dW_t$

Differential laws are the grammar of change. They govern everything from trajectories to uncertainty. The heat equation is the law of motion for probability itself.