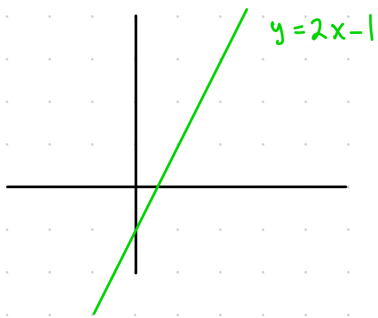
Calculus Review

Topics:
- Overview of derivatives & integrals
- Techniques / Applications of both
- Parametric equations and polar coordinates
- Vectors - operations & functions
- Multivariable calculus - partial derivatives, multiple integrals, vector calculus

Derivatives & Integrals

* The heart of calculus is about being able to understand and model change
↳ derivatives & integrals are core tools/concepts for doing this.

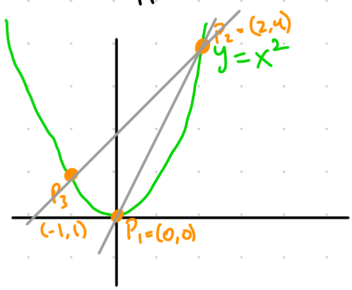
From algebra, change was described using slope



Equation of line - $y=mx+b$
slope - m describes rate of change of output as we vary input
$m=2 \to$ for every 1 unit change in x (input) we see 2 unit change in y (output)

Slope Formula $\to$ $m=\frac{y_2-y_1}{x_2-x_1}$ Describes average change over some interval

What happens when function becomes a little more complicated?



We need to adjust how we describe change.
Avg change can still be used

$$m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{4-0}{2-0}=2$$

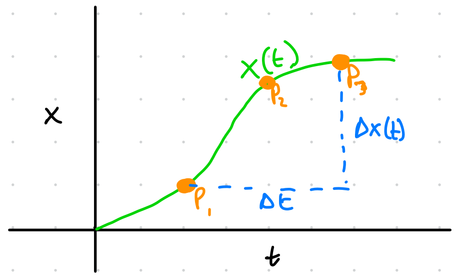
From $P_1 = (0,0)$ to $P_2 (2,4)$ output changes 2 units for every 1 unit change in input.

What about from P_3 to P_2? Slope in graph above looks different

$$m_2 = \frac{4-1}{2-(-1)} = \frac{3}{3} = 1$$

When we choose different interval, avg change is different.
How do we obtain more exact/useful value for change?

Ex. Car driving over time

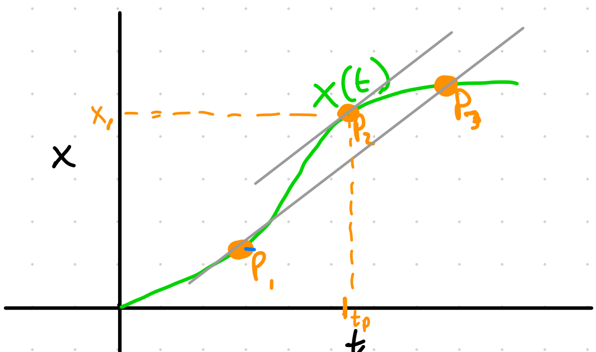


Position x of car at point in time t is described by function x(t)

3 points of interest marked on graph

Change of position over time interval or average change is simple enough to calculate

What if I want to know the rate at which x(t) is changing at a single point in time?



Similar to calculating change over interval (eg. $P_1 \to P_3$)
We just make interval infinitely small

What is rate of change of x(t) at P_2

$$\lim_{\Delta t \to 0} \frac{x(t_p + \Delta t) - x(t_p)}{\Delta t}$$

new position (y_2)

initial position (y_1)

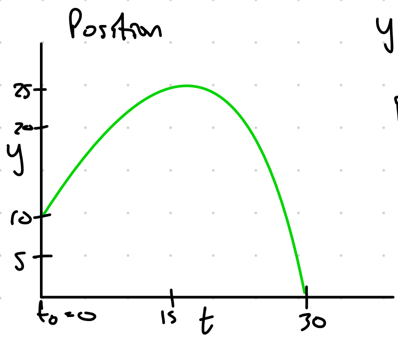
← change in time $(x_2 - x_1)$

To find this instantaneous change at point in time, we look at time interval infinitely small (approaching 0)

↳ This type of change is the derivative

Derivative gives us rate of change at any point on continuous curve

Position of ball thrown straight up & falling back to ground



$$y = y_0 + V_0 t + \frac{1}{2}at^2$$

Derivative allows us to see rate position is changing at any point in time t

initial height $y_0 = 10m$
initial velocity $V_0 = 3 m/s$
acceleration (gravity) $a = -9.8 m/s^2$

Derivative of position w/ respect to time gives velocity

$$\frac{dy}{dt} = V(t) = V + at$$

Derivative of velocity w/ respect to time gives acceleration

$$\frac{dv}{dt} = a(t) = a$$

Fundamental way to calculate derivative is w/ power formula

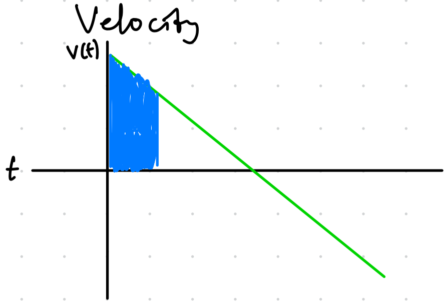
$$\frac{d}{dx}[x^n] = n x^{n-1}$$ applied to every term in function

eg. $y = 2x^3 - 3x^2 - 4x$ $\frac{dy}{dx} = 6x^2 - 6x - 4$

Integral gives total change over an interval
known as antiderivative

If $\frac{dx}{dt} = v(t)$ at $t=1$ gives rate of change of position (velocity)
at $t=1$, then $\int_0^3 v(t)dt$ gives total change of position, Δx, on

time interval [0,3)



Derivative finds slope at particular point

Integral gives us area under the curve

Calculating and Applying Derivatives

For basic polynomials we can use power rule

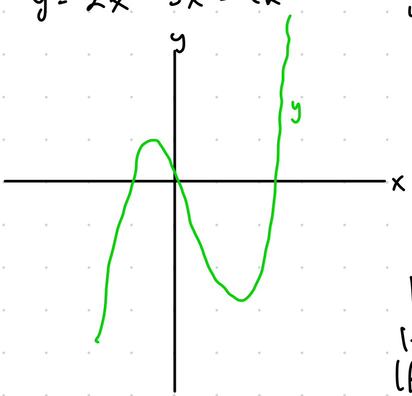
$y = 6x^2 - 4x + 2 \longrightarrow y' = 12x - 4$

$f(x) = x^4 + 2x^3 - 4 \longrightarrow f'(x) = 4x^3 + 6x^2$

$y = x^3 + 2x \longrightarrow \frac{dy}{dx} = 3x^2 + 2$

Useful application of derivative is understanding shape of graph

$y = 2x^3 - 3x^2 - 4x$ $\qquad y' = 6x^2 - 6x - 4$

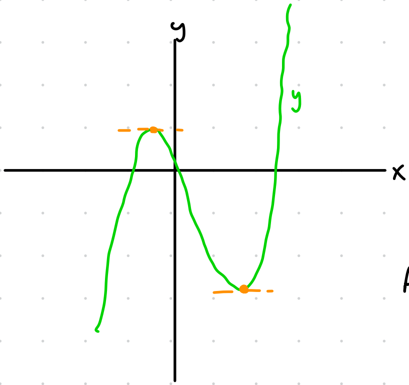


y' gives us rate of change at any x, so we can use it to know if function is increasing, decreasing, at a critical point, or asymptote

If $y' > 0$ @ x then y is increasing at x
If $y' < 0$ @ x then y is decreasing
If $y' = 0$ @ x then y is at critical point (local max or min)
If $y' =$ undefined @ x, then y is at asymptote

Useful to find local min & max of function

↳ start by finding critical points ($y' = 0$) → ie where tangent is horizontal



$y' = 6x^2 - 6x - 4$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{6 \pm \sqrt{36 - 4(6)(-4)}}{12}$$

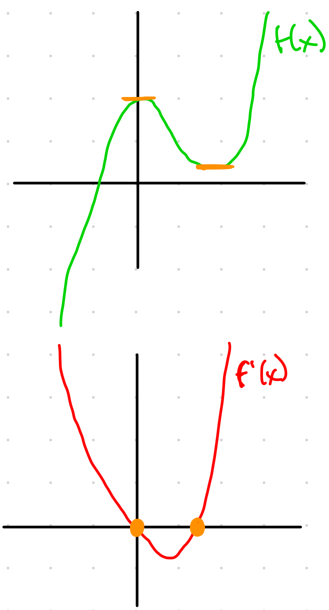
$$= \frac{6 \pm \sqrt{132}}{12}$$

$$x = \frac{3 \pm \sqrt{33}}{6}$$

Not the clearest example but it works

Assuming those zeros are correct, those are the points y changes direction

Consider another function $f(x) = x^3 - 2x^2 + 2$



$f'(x) = 3x^2 - 4x$

$3x^2 - 4x = 0$

$x(3x - 4) = 0$

$x = 0, \frac{4}{3}$

Check points around critical points to determine local minimum or maximum

X	0	4/3
X−	$f'(-1) = 7$ +	$f'(1) = -1$ −
X+	$f'(.5) = -1.25$ −	$f'(2) = 4$ +

$f'(x)$ graph shows $f'(x) > 0$, ie $f(x)$ ↑ until $x = 0$, then $f'(x)$ becomes negative

left of 0 is increasing, right of zero is decreasing

$x = 0$ → local maximum

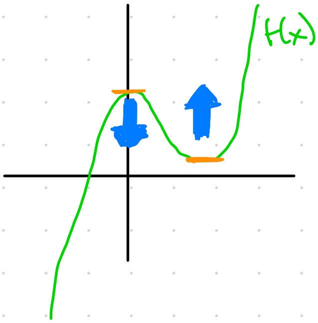
left of $4/3$ is decreasing & right is increasing

$x = 4/3$ → local minimum

double derivative gives concavity at particular point

$\cup$ ← concave up $\quad\quad$ $\cap$ ← concave down

If $y'' < 0$ @ x, then y is concave down at x
if $y'' > 0$ @ x, then y is concave up



$f''(x) = 6x - 4$

$f''(0) = 6(0) - 4 = -4$ → concave down
$f''(4/3) = 6(4/3) - 4 = 4$ → concave up

Basic formula for calculating integral is reverse of power rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$ ← constant of integration

Indefinite integral → result is function

eg. $\int x^2 + 2\,dx = \frac{x^3}{3} + 2x + c$

Definite Integral → result is number

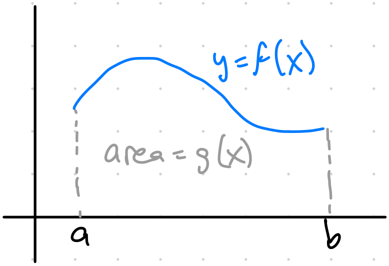
eg. $\int_0^3 x^2 + 2\,dx = \frac{x^3}{3} + 2x \Big|_0^3 = \frac{(3)^3}{3} + 2(3) - \cancel{\left[\frac{0^3}{3} + 2(0)\right]}^{0}$

$= 15$

→ area under curve on specific interval

This brings us to **Fundamental Theorem of Calculus**

↳ essentially describing inverse relationship between differential & integral calculus



$$g(x) = \int_a^b f(x)\,dx$$

$$g'(x) = f(x)$$

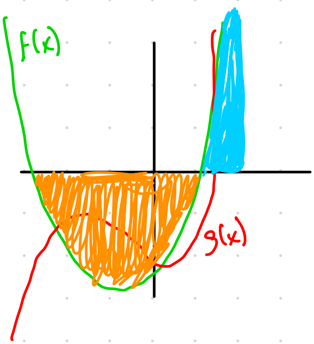
also...

Calculating area under curve on interval [a,b] →

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$F'(x) = f(x)$$

$$f(x) = x^2 + 2x - 4$$



$$g(x) = \int f(x)\,dx = \frac{x^3}{3} + x^2 - 4x$$

$$\int_1^3 f(x)\,dx = \frac{x^3}{3} + x^2 - 4x \Big|_1^3$$

$$= \frac{27}{3} + 9 - 12 - \left[\frac{1}{3} + 1 - 4\right]$$

$$= 6 - \left(-\frac{8}{3}\right)$$

$$= \frac{26}{3}$$

$$\int_{-3}^1 f(x)\,dx = \frac{x^3}{3} + x^2 - 4x \Big|_{-3}^1$$

$$= \frac{1}{3} + 1 - 4 - \left[\frac{(-3)^3}{3} + (-3)^2 - 4(-3)\right]$$

$$= -14.667$$

the resulting value is net change of g(x) over interval [a,b]

Methods for solving derivatives

Chain Rule:

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$

$y = (2x^2-4)^3 \quad g(x) = 2x^2-4 \quad f(x) = x^3 \quad g'(x) = 4x$

$f'(x) = 3x^2 \quad f'(g(x)) = 3(2x^2-4)^2 \quad f'(g(x))\,g'(x) = 3(2x-4)^2 \cdot 4x$

$y' = 12x(2x-4)^2$

Implicit Differentiation:
If function is not explicitly defined $y=x$, utilize chain rule

eg. $x^2+y^2=4$

differentiate both sides w/ respect to x

$$\frac{d}{dx}[x^2+y^2] = \frac{d}{dx}[4]$$

$$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = 0$$

chain rule

$$2x + 2y\frac{dy}{dx} = 0$$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$
Quotient Rule:

* Integration Methods:

Substitution Rule: Inverse of chain rule

Use when integral has form $\int f(g(x))\,g'(x)\,dx$

set $g(x)=u \rightarrow \int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

eg $\int 2x\sqrt{1+x^2}\,dx$

$g(x) = 1+x^2$
$g'(x) = 2x\,dx$ which is part of integrand

$u = 1+x^2$
$du = 2x\,dx$

then plug u back in

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

eg. $\int \cos(1+5t)\,dt$

$u = 1+5t$
$du = 5\,dt$
$du/5 = dt$

coefficient can be factored out of integrand

$$\Rightarrow \int \cos(u)\frac{1}{5}\,du$$
$$= \frac{1}{5}\int \cos(u)\,du$$
$$= \frac{1}{5}\sin(u) + c = \boxed{\frac{1}{5}\sin(1+5t) + c}$$

Integration by Parts: Inverse of product rule from differentiation

Derivation.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\int f(x)g'(x) + g(x)f'(x)\,dx = f(x)g(x)$$

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Let $u = f(x)$ $dv = g'(x)\,dx$, then $du = f'(x)\,dx$ & $v = g(x)$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$\int e^x \sin x\,dx$ neither becomes simpler when differentiated so doesn't matter

$u = e^x$ $dv = \sin x\,dx$
$du = e^x\,dx$ $v = -\cos x$

$$\Rightarrow -e^x\cos x + \int e^x \cos x\,dx$$

Perform int. by parts again

$u = e^x$ $dv = \cos x\,dx$
$du = e^x\,dx$ $v = \sin x$

$$-e^x\cos x + \left[e^x\sin x - \int e^x\sin x\,dx\right]$$

IMPORTANT TECHNIQUE

neither e^x nor $\sin x$ will be reduced w/ differentiation
so this process could go on for a while

However, if we set up full expression

$$\underline{\int e^x\sin x\,dx} = -e^x\cos x + e^x\sin x - \underline{\int e^x\sin x\,dx}$$

the integral we want to solve for appears on both sides.
we can set up equation to be solved for unknown integral

$$2\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x$$

$$\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x)\,dx$$

* Cover trig integrals, trig substitution, rational fractions

* Look over <u>IMPROPER INTEGRALS</u> → convergence

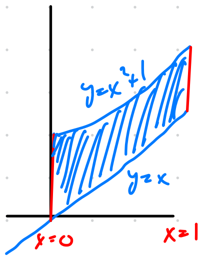
Applications of Integrals

Area between curves → Find area of region between graph of 2 functions

In general, area A of region bounded by $y=f(x)$, $y=g(x)$ & lines $x=a$, $x=b$, where f & g are continuous & $f(x)\geq g(x)$ for all x in $[a,b]$

$$A=\int_a^b [f(x)-g(x)]\,dx$$

Find area of region bounded above by $y=x^2+1$ & below by $y=x$ between $x=0$ & $x=1$



$$A=\int_a^b [f(x)-g(x)]dx$$

$$=\int_a^b f(x)dx-\int_a^b g(x)dx$$

$$=\frac{x^3}{3}+x\Big|_0^1-\left[\frac{x^2}{2}\Big|_0^1\right]$$

$$=\frac{x^3}{3}-\frac{x^2}{2}+x\Big|_0^1$$

$$=\frac{1}{3}-\frac{1}{2}+1-[0]=\boxed{\frac{5}{6}}$$

Oftentimes you will be given 2 curves only & have to find intersection points

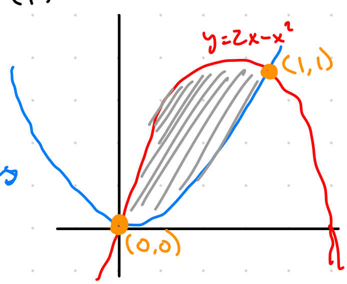
Find area of region enclosed by $y=x^2$ & $y=2x-x^2$

Find points where 2 graphs intersect (i.e. equal each other)

$$2x-x^2=x^2$$
$$2x-2x^2=0$$
$$2x(x-1)=0$$

$$x=0,\ 1$$
$$y=(0)^2=0=2(0)-(0)^2$$
$$y=(1)^2=1=2(1)-(1)^2$$

Intersection points → $(0,0)$ & $(1,1)$



*Be sure to identify upper & lower boundary
$f(x)=2x-x^2$ b/c $f(x)\geq g(x)$ for all x on interval

$$A=\int_a^b [f(x)-g(x)]dx$$

$$=\int_a^b 2x-x^2-x^2\,dx$$

$$=\int_0^1 2x-2x^2dx=x^2-\frac{2}{3}x^3\Big|_0^1=1-\frac{2}{3}-[0]=\boxed{\frac{1}{3}}$$

If we had chosen incorrect boundaries

$f(x)=x^2 \quad g(x)=2x-x^2$

$$\int_0^1 x^2-[2x-x^2]\,dx = \int_0^1 2x^2-2x\,dx = \frac{2}{3}x^3-x^2\Big|_0^1 = \frac{2}{3}-1 = -\frac{1}{3}$$

*negative area doesn't make sense so we should catch that mistake

Few variations to explore, namely writing x as function of y & calculating area using

$x=f(y),\ x=g(y)\quad y=c,\ y=d$

Volume

$V=\int_a^b A(x)\,dx$ explore in detail

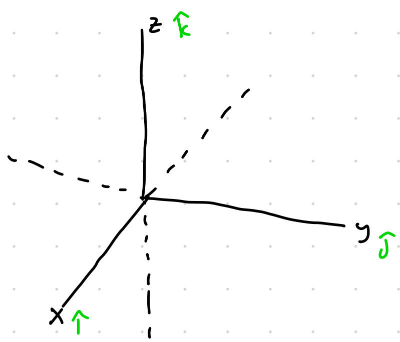
Volumes of cylindrical shells

* Ch 8 – Arc Length, Area of surface of revolution, etc

* Ch 10 – Parametric Equations & polar coordinates

Vectors & 3D Space

Expanding coordinate system from 2d → 3d, adding z axis

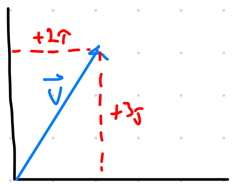


$\hat{i}, \hat{j}, \hat{k}$ → unit vectors for x, y, z axes

A vector is a mathematical object w/ magnitude & direction.
Often represented as arrow

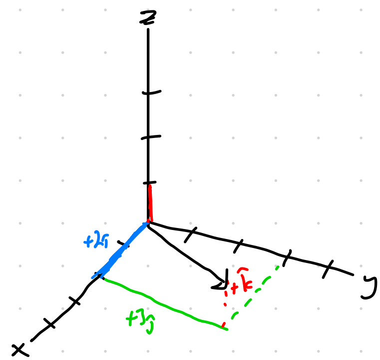
A vector in n-d space will have n components

eg $\vec{V} = 2\hat{\imath} + 3\hat{\jmath}$ ← +2 units $\hat{\imath}$ direction (x-axis) & +3 units $\hat{\jmath}$ direction (y-axis)



$\vec{V} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$

Can also be represented as coordinate point
$\vec{V} = (2, 3, 1)$
x y z



Vector arithmetic & scaler multiplication should be familiar by now

Dot Product (aka Inner Product)

matrix multiplication & change of bases

explore geometric interpretations, specifically in Linear Algebra

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Produces scalar value

* Useful in exploring angle between 2 vectors

$$a \cdot b = |a||b| \cos \theta$$

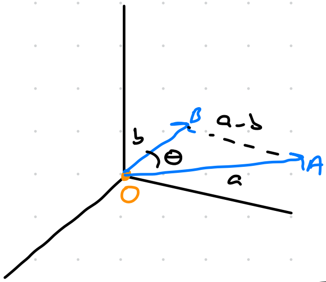
$\therefore$

$$\cos \theta = \frac{a \cdot b}{|a||b|}$$

More generally, inner product is used to analyze similarity between 2 mathematical objects eg (how similar are 2 sound waves)

$|a|$ → magnitude of vector a

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}$$ in n-dimensional space



If $\theta = 0$ or $\theta = \pi$, the 2 vectors are parallel

* 2 vectors a & b are orthogonal (perpendicular) iff $a \cdot b = 0$

if $a \cdot b = 0$, implies $\cos\theta = 0$ so $\theta = \frac{\pi}{2}$ (90°)

eg. Find angle θ between $a = \langle 2, 2, -1\rangle$ & $b = \langle 5, -3, 2\rangle$

$|a| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$ $\quad$ $|b| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$

$a \cdot b = 2(5) + 2(-3) + (-1)(2) = 2$

$\cos\theta = \frac{2}{3\sqrt{38}}$

$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46$ (~84°)

Show $2\hat{i} + 2\hat{j} - k$ is perpendicular to $5\hat{i} - 4\hat{j} + 2\hat{k}$

$(2\hat{i} + 2\hat{j} - k) \cdot (5\hat{i} - 4\hat{j} + 2\hat{k}) = (2)(5) + (2)(-4) + (-1)(2) = 0$

dot product between 2 vectors is 0 ⇒ orthogonal